# Charity Auctions for the Happy Few

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# What is a charity auction ?

Contrary to non-charity auctions, here the amount paid is "never lost"
An investor, who bought a Dior perfume for 60 000 yuans said in the *Beijing Review*:

> *I would never buy perfume for this amount normally, but this time it is for charity. I feel very happy.*

Auctions are increasingly used as a means to raise money for charitable purposes.
Examples: *Yahoo!*, *Giving Works* of *eBay*, *cMarket Charitable Auctions Online* (2002) or *Pass It On Celebrity Charity Auction* (2003)

# Literature

Theoritical papers: Ettinger (2002), Maasland and Onderstal (ET, 2007), Goeree, Maasland, Onderstal and Turner (JPE, 2005), Engers and McManus (IER,2007)

Experimental papers: Orzen (2005), Carpenter, Homes and Matthews (EJ, 2008), Onderstal and Schram (IER, 2008)

# Motivation

The purpose of this paper is to determine whether or not all-pay auctions can raise higher revenue for charity when the asymmetry between bidders is strong.

A lot of charity auctions are conducted among rich people during charity dinners. Potential bidders are acquaintances or know one another well. A complete information environment is well suited.

# Allocation Rule

An object is sold and allocated to one of the potential bidders $N = \{1, ..., n\}$ contingent upon their bids $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$.

An all-pay auction is a pair $(\boldsymbol{a}, \boldsymbol{t})$

$\boldsymbol{a} = (a_1, ..., a_n) : \mathbb{R}^n_+ \longrightarrow [0, 1]^n$ is such that the winner $i$ gets the object iff $a_i(x) = 1$ given the bids and $\sum_{i=1}^n a_i(x) = 1$ for all $x$.

$$\begin{cases} a_i(x) = \frac{1}{\#Q(\boldsymbol{x})} \text{ if } i \in Q(\boldsymbol{x}) \\ a_i(x) = 0 \text{ otherwise} \end{cases}$$

where $Q(\boldsymbol{x}) := \{j | j = \arg\max\{x_k, k \in N\}\}$.

# Payment Rule

$\boldsymbol{t} = (t_1, ..., t_n) : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$
In the first-price all-pay auction, each bidder pays his own bid

$$t_i(x) = x_i \;\; \forall i \in N$$

In the second-price all-pay auction the winner pays the second highest bid and the losers their own bid

$$t_i(\boldsymbol{x}) = x^{(2)} \text{ if } i \in Q(x)$$
$$t_i(\boldsymbol{x}) = x_i \text{ otherwise}$$

with $x^{(2)}$ the second order statistic of sample $(x_1, ..., x_n)$.

Denote $h_i(\boldsymbol{t}(\boldsymbol{x}))$ the externality that the bidder $i$ takes advantage of such that

$$h_i(\boldsymbol{t}(\boldsymbol{x})) = h_i\left(\sum_{j=1}^{n} t_j(\boldsymbol{x})\right) = \alpha_i \sum_{j=1}^{n} t_j(\boldsymbol{x})$$

where $\alpha_i \geq 0$.
The bidder $i$'s utility is given by

$$U_i(\boldsymbol{x}) = \tilde{U}_i(a_i, \boldsymbol{t}) = v_i a_i(\boldsymbol{x}) - t_i(\boldsymbol{x}) + \alpha_i \sum_{j=1}^{n} t_j(\boldsymbol{x})$$

# Assumptions

## Assumption 1 ($A1$)

*$\tilde{U}_i(a_i, \boldsymbol{t})$ is a continuous and differentiable function in the transfer functions $t_j$ for all $j$.*

## Assumption 2 ($A2$)

$$\forall x_i \geq 0 \;\; \frac{\partial \tilde{U}_i}{\partial t_i(\boldsymbol{x})}(a_i, \boldsymbol{t}) < 0 \text{ equivalent to } \alpha_i \sum_{j=1}^{n} \frac{dt_j(\boldsymbol{x})}{dt_i(\boldsymbol{x})} < 1.$$

# Expected Utility

$$\mathbb{E}U_i(x_i, X_{-i}) = v_i \prod_{j \neq i} F_j(x_j) - \int_{\mathbb{R}_+^{n-1}} t_i(x) \prod_{j \neq i} dF_j(x_j)$$
$$+ \alpha_i \int_{\mathbb{R}_+^{n-1}} \sum_{j=1}^{n} t_j(x) \prod_{j \neq i} dF_j(x_j)$$

with $X_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$

$\frac{v_1}{1-\alpha_1} > \frac{v_2}{1-\alpha_2} > ... > \frac{v_n}{1-\alpha_n}$ is common knowledge.

# Equilibrium in the First-Price All-Pay Auction

Assumption $A2$ implies that $\alpha_i < 1$
A bidder takes part in the auction if $\mathbb{E}U_i(x, X_{-i}) \geq \alpha_i \sum_{j \neq i} \mathbb{E}X_j$

Bidder $i$'s expected utility with $n$ potential competitors is given by

$$\mathbb{E}U_i(x_i, X_{-i}) = \Pi_{j \neq i}^{n} F_j(x_j) v_i - (1 - \alpha_i) x_i + \alpha_i \sum_{j \neq i} \mathbb{E}X_j$$

## Proposition 1

*There is a unique Nash equilibrium and the mixed strategies are*

$$F_1(x) = \frac{1-\alpha_2}{v_2} x \ \ \forall x \in \left[0, \frac{v_2}{1-\alpha_2}\right]$$

$$F_2(x) = 1 - \frac{1-\alpha_1}{1-\alpha_2}\frac{v_2}{v_1} + \frac{1-\alpha_1}{v_1} x \ \ \forall x \in \left(0, \frac{v_2}{1-\alpha_2}\right]$$

*All others bidders use the pure strategy of zero and do not take part in the auction:* $F_j(0) = 1$ *for* $j \in \{3, ..., n\}$. *The expected revenue is given by* $\mathbb{E}R = \frac{1}{2}\frac{v_2}{1-\alpha_2}\left(\frac{1-\alpha_1}{1-\alpha_2}\frac{v_2}{v_1} + 1\right)$.

## Corollary 1

*All bidders obtain a positive payoff. Indeed, the bidder with the two highest adjusted-value obtains a positive payoff* $U_1^\star = v_1 - \frac{1-\alpha_1}{1-\alpha_2} v_2 + \frac{\alpha_1}{2}\frac{1-\alpha_1}{v_1}\left(\frac{v_2}{1-\alpha_2}\right)^2$ *and* $U_2^\star = \frac{v_2}{2}\frac{\alpha_2}{1-\alpha_2}$ *and their competitors get* $U_i^\star = \alpha_i \frac{v_2}{1-\alpha_2}\left(\frac{1-\alpha_1}{v_1}\frac{v_2}{1-\alpha_1} + 1\right)$ *for* $i \in \{3, ..., n\}$.

# Non-linear Externalities

$$\mathbb{E}U_1(x_1, X_2) = F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2}h_1(x_1, X_2)$$
$$\mathbb{E}U_2(x_2, X_1) = F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1}h_2(X_1, x_2)$$

Details

Bidder $i$ takes part to the auction if $\mathbb{E}U_i(x_i, X_j) \geq \mathbb{E}_{X_j}h_i(0, X_j)$

## Proposition 2

*Given $A1 - A2$ and given that the two bidders have a common support $[0, b]$, the mixed strategy equilibrium exists.*

Sketch of the proof

# Equilibrium in the Second-Price All-Pay Auction

$X_i \subseteq [0, +\infty)$, assumption $A2$ allows us to write that $\alpha_i < 1/2$

## Proposition 3

*Only two bidders among n participate actively to the auction. Their mixed strategies are given by an exponential distribution defined as follows,*

$$F_i \sim \mathcal{E}\left(\frac{1-2\alpha_j}{v_j}\right) \text{ and } \mathbb{E}R = \frac{2v_i v_j}{(1-2\alpha_i)v_j + (1-2\alpha_j)v_i}$$

For $n = 2$

$$\mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1-2\alpha_i)x)dF_j(x) - (1-2\alpha_i)x_i(1-F_j(x_i))$$

For $n > 2$
Note $G_i(x) = \prod_{j \neq i} F_j(x)$.

$$\begin{aligned}\mathbb{E}U_i(x_i, X_{-i}) = &\int_0^{x_i} (v_i - (1-\alpha_i)x_i)dG_i(x_i) - (1-\alpha_i)x_i(1-G_i(x_i)) \\ &+ \alpha_i \sum_{l \neq i} \int_{\mathbb{R}_+} x_l \left(1 - \mathbb{1}_{x_i \leq x_l} \prod_{k \neq l,i} F_k(x_l)\right) dF_l(x_l) \\ &+ \alpha_i \sum_{l \neq i} \Bigg( \int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k \neq l,i} x_k \prod_{\substack{m \neq i,k,l \\ k \neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) \\ &+ x_i \prod_{m \neq i,l} F_m(x_i)(1 - F_l(x_i)) \Bigg)\end{aligned}$$

$$\mathbb{E}U_i(x_i, X_{-i}) = v_i \prod_{j \neq i} F_j(x_j) - (1-\alpha_i) \int_{\mathbb{R}_+^{n-1}} t_i(x) \prod_{j \neq i} dF_j(x_j)$$
$$+ \alpha_i \int_{\mathbb{R}_+^{n-1}} \sum_{\substack{l=1 \\ l \neq i}}^{n} t_l(x) \prod_{j \neq i} dF_j(x_j)$$

$$\sum_{l \neq i} \int_{\mathbb{R}_+^{n-1}} t_l(\boldsymbol{x}) \prod_{j \neq i} dF_j(x_j) = \sum_{l \neq i} \Bigg\{ \underbrace{\int_{\mathbb{R}_+^{n-1}} x_l \mathbb{1}_{\substack{\exists k / x_l < x_k \\ k \neq l}} \prod_{j \neq i} dF_j(x_j)}_{B_I}$$
$$+ \underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{\substack{x_l \geq x_k \\ \forall k \neq l}} \prod_{j \neq i} dF_j(x_j)}_{B_{II}} \Bigg\}$$

# Summary

| $v_1 > v_2 > v_3 > v_i \ \forall i > 3$ | $R^{WP1}$ | $R^{WP2}$ | $\mathbb{E}R^{AP1}$ | $\mathbb{E}R^{AP2}$ |
|---|---|---|---|---|
| $\alpha > 0$ | $v_2$ | $v_1$ | $\frac{1}{2}\frac{v_2}{1-\alpha}\left(\frac{v_2}{v_1}+1\right)$ | $\frac{2}{1-2\alpha}\frac{v_1 v_i}{v_1+v_i}, \ i \neq 1$ |
| $\alpha = 0$ | $v_2$ | $v_2$ | $\frac{v_2}{2}\left(\frac{v_2}{v_1}+1\right)$ | $\frac{v_1 v_i}{v_1+v_i}, \ i \neq 1$ |

Table: Revenues and expected revenues for every design

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Table: Revenues and expected revenues for every design

## Definition

*The level of asymmetry between bidders' valuations will be considered very high if* $v_1 - v_2 > 2\alpha(v_1 + v_2)$, *high if* $v_1 - v_2 > 2\alpha v_1$, *low if* $v_1 - v_2 < 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$ *and medium if* $2\alpha v_1 > v_1 - v_2 > 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$.

# Revenue Comparisons

## Proposition 4

*We assume that $\alpha_i = \alpha \; \forall i$ and that the bidder with the highest adjusted-value takes part in the second-price all-pay auction. Then, this design gives the highest revenues:*

$$\mathbb{E}R^{AP2} > R^{WP2} \text{ iff the level of asymmetry is not very high,}$$

$$\mathbb{E}R^{AP2} > R^{WP1} \text{ and } \mathbb{E}R^{AP2} > \mathbb{E}R^{AP1}$$

*All other things being equal, $\mathbb{E}R^{AP1} > R^{WP2}$ iff the level of asymmetry between valuations is low, $R^{WP2} > \mathbb{E}R^{AP1} > R^{WP1}$ iff this level is medium, and $R^{WP1} > \mathbb{E}R^{AP1}$ iff it is high.*

Given $v_1$ and $v_2$, first-price all-pay auction is dominated by first and second-price winner-pay auctions when the bidders' altruism level is less than $\frac{1}{2}(1 - \frac{v_2}{v_1})$ and only by second-price winner-pay auction when it is less $1 - \frac{1}{2}\frac{v_2}{v_1}(\frac{v_2}{v_1} + 1)$ and superior to $\frac{1}{2}(1 - \frac{v_2}{v_1})$.

# Illustration

From left to right, $\frac{v_2}{v_1}$ varies from 0.9 to its limit in zero with a 0.1 step.

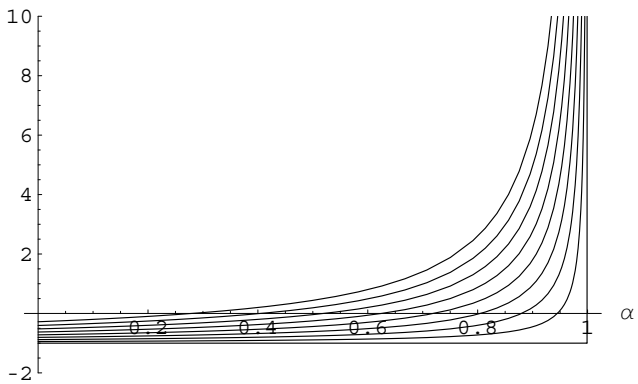



Figure: $\mathbb{E}R^{AP1} > R^{WP2}$

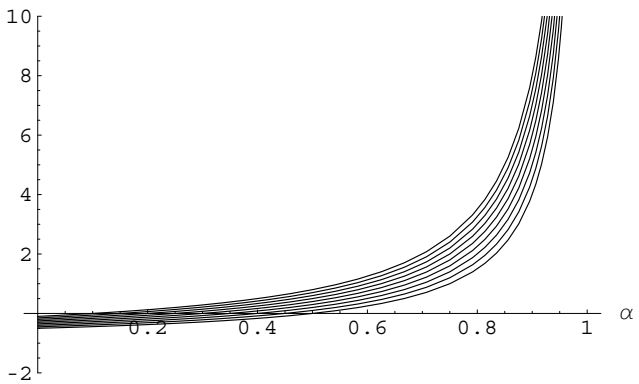
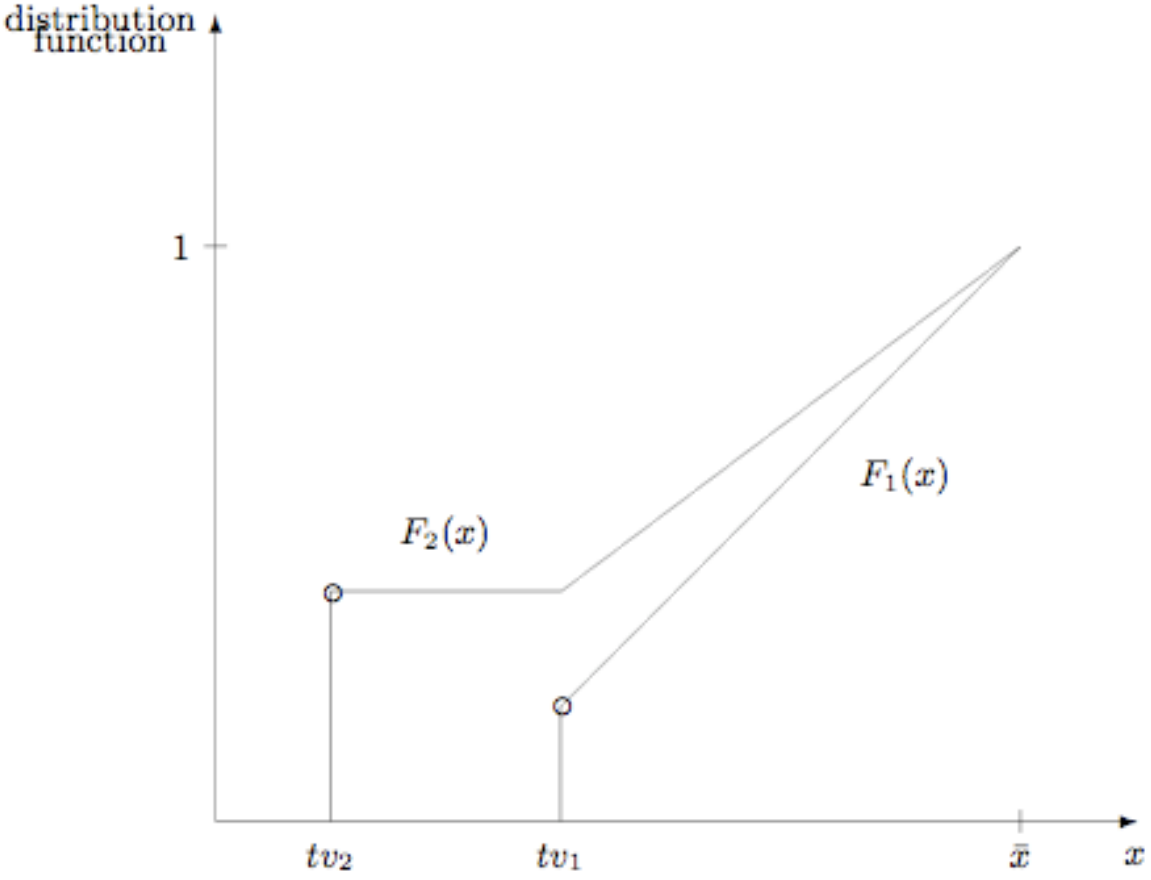



Figure: $\mathbb{E}R^{AP1} > R^{WP1}$

# Equilibrium




detail

# Expected Revenue

## Proposition 5

*Imposing a minimal bid to every bidder permits to improve the first-price all-pay auction's efficiency compared to the first-price winner-pay auction. There is a threshold $t$ above which the all-pay auction dominates the winner-pay auction when the values' asymmetry is considered high.*

# Expected Revenue: Illustration

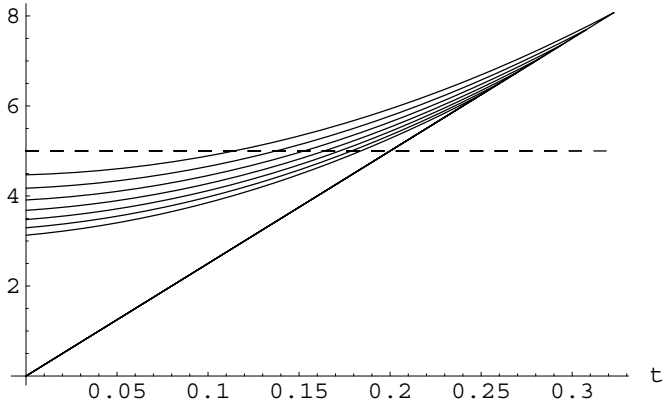



Figure: Expected revenue with a high asymmetry

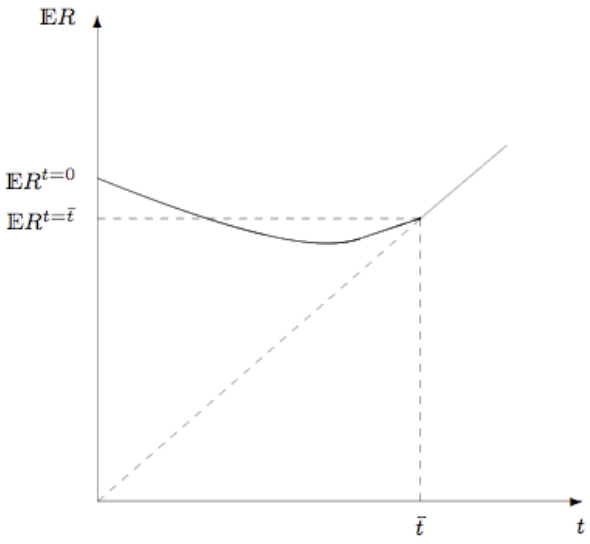



Figure 1: $\mathrm{E}R$ for $\alpha > \max\{\frac{v_1}{v_1+v_2}, \frac{v_1-v_2}{2v_1}\}$

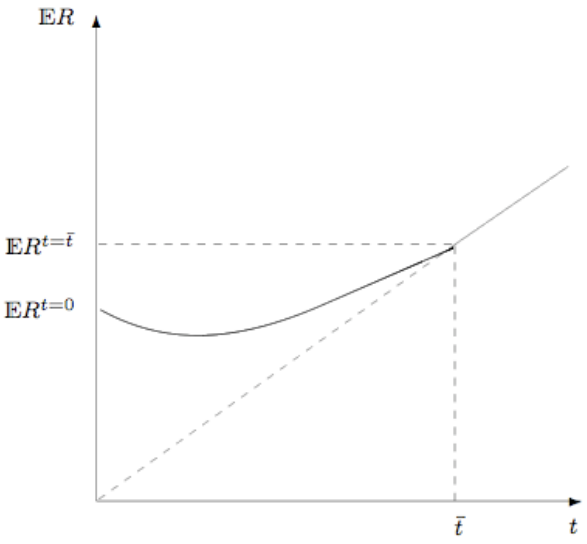



Figure 2: $\mathrm{E}R$ for $\frac{v_1}{v_1+v_2} > \alpha > \frac{v_1-v_2}{2v_1}$

Thank you for your attention!

$$\begin{aligned}
\mathbb{E}U_1(x_1, X_2) &= F_2(x_1)\left(v_1 + \mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) - x_1\right) \\
&\quad + (1 - F_2(x_1))(\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \geq x_1) - x_1) \\
\mathbb{E}U_2(x_2, X_1) &= F_1(x_2)\left(v_2 + \mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \leq x_2) - x_2\right) \\
&\quad + (1 - F_1(x_2))(\mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \geq x_2) - x_2)
\end{aligned}$$

with $\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) = \dfrac{1}{F_2(x_1)} \displaystyle\int_0^{x_1} h_1(x_1, x_2) dF_2(x_2)$

It can also be written as

$$\begin{aligned}
\mathbb{E}U_1(x_1, X_2) &= F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2} h_1(x_1, X_2) \\
\mathbb{E}U_2(x_2, X_1) &= F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1} h_2(X_1, x_2)
\end{aligned}$$

back

$T : F(x) \longmapsto TF(x)$ and

$$TF(x) \equiv \lambda x - \lambda \int_0^b h(x,y) f(y) dy + \text{ constant} \tag{1}$$

The set $D = \{F \in C[0,b] \backslash ||F|| \leq 1\}$, which includes all the continuous distribution functions, is closed and convex but not compact. Thus, to prove that (1) has a solution, we apply the following Schauder's second theorem:

### Theorem (Schauder, 1930)

*If $D$ is a closed convex subset of a normed space and $E$ a relatively compact subset of $D$, then every continuous mapping of $D$ to $E$ has a fixed-point.*

To apply this theorem, we need to prove two parts.

- $T(D) \equiv E = \{TF \backslash F \in D\}$ is relatively compact which is equivalent to prove $E$ is uniformly bounded and equicontinuous;
- Second, $T$ is a continuous mapping from $D$ to $E$.

back

$$\mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1-2\alpha_i)x)dF_j(x) - (1-2\alpha_i)x_i(1-F_j(x_i))$$

Note $G_i(x) = \prod_{j\neq i} F_j(x)$.

$$\begin{aligned}\mathbb{E}U_i(x_i, X_{-i}) = &\int_0^{x_i} (v_i - (1-\alpha_i)x)dG_i(x) - (1-\alpha_i)x_i(1-G_i(x_i)) \\ &+ \alpha_i \sum_{l\neq i} \int_{\mathbb{R}_+} x_l \left(1 - \mathbb{1}_{x_i \leq x_l} \prod_{k\neq l,i} F_k(x_l)\right) dF_l(x_l) \\ &+ \alpha_i \sum_{l\neq i} \left( \int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k\neq l,i} x_k \prod_{\substack{m\neq i,k,l \\ k\neq l}} F_m(x_k) dF_k(x_k) dF_l(x_l) \right. \\ &\left. + x_i \prod_{m\neq i,l} F_m(x_i)(1-F_l(x_i)) \right)\end{aligned}$$

back

# Equilibrium

## Proposition 6

*Given the bidders' adjusted-values,* $(1-\alpha_1 t)\tilde{x}_1$ *and* $(1-\alpha_2 t)\tilde{x}_2$*, there is a unique Nash equilibrium. The bidders' strategies for all* $x \in (tv_1; \bar{x}]$ *are*

$$F_1(x) = \alpha_2 t + \frac{x}{\tilde{x}_2} \text{ and } F_2(x) = 1 + \frac{x - \bar{x}}{\tilde{x}_1}.$$

*Every bidder has one mass point: it is* $tv_1$ *for bidder 1 and* $tv_2$ *for bidder 2.*

A bidder's decision is given by her probability to participate,

$$1 - F_1(tv_1) = 1 - \alpha_2 t - \frac{tv_1}{\tilde{x}_2} \text{ and } 1 - F_2(tv_2) = \frac{\bar{x} - tv_1}{\tilde{x}_1}$$

# Expected Revenue

Additionally, if the maximum bid $\bar{x}$ is inferior to bidder 1's minimum bid $\bar{x} \leq tv_1$, offering a higher bid than their minimum bid is dominated for all bidders. Hence, $\mathbb{E}R = t(v_1 + v_2)$ for all $t \geq \bar{t}$ where $\bar{t} \equiv \dfrac{\tilde{x}_2}{v_1 + \alpha_2 \tilde{x}_2}$.

## Proposition 7

*Given the distribution functions $F_1(.), F_2(.)$ at equilibrium, the expected revenue raised for charity is*

$$\begin{cases} \bar{x}^2 \dfrac{\tilde{x}_1 + \tilde{x}_2}{2\tilde{x}_1\tilde{x}_2} + (tv_1)^2 \dfrac{\tilde{x}_1 - \tilde{x}_2}{2\tilde{x}_1\tilde{x}_2} + t^2 v_1 \alpha_2 + tv_2 \left(1 + \dfrac{tv_1 - \bar{x}}{\tilde{x}_1}\right) & \text{if } t < \bar{t} \\ t(v_1 + v_2) & \text{otherwise} \end{cases}$$

with $\bar{t} \equiv \dfrac{\tilde{x}_2}{v_1 + \alpha_2 \tilde{x}_2}$ back